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MINIMAL LAGRANGIAN DIFFEOMORPHISMS AND THE MONGE-AMPÈRE EQUATION

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0. Introduction

In this paper we consider two problems: one from geometry, one from analysis.

Consider, here and throughout this paper, two connected, simply connected, closed, bounded domains D_1 and D_2 in $\mathbb{R}^2$ with smooth boundaries. Suppose that the domains have equal area. It is well-known that there exists an area-preserving diffeomorphism $\psi : D_1 \to D_2$ which is smooth up to the boundary. (For a discussion of this and related questions see [**?**]). However, the differential equations which determine ψ form an underdetermined system and hence ψ cannot be expected to closely reflect the geometry of the domains D_1 and D_2. Consequently, it is an interesting problem to find further conditions on an area preserving diffeomorphism to more tightly link the diffeomorphism to the geometry of the domains.

Such a condition is suggested by the following theorem of R. Schoen [**?**] and, independently, F. Labourie [**?**]. Let M be a compact Riemann surface of genus $g \geq 2$. Let g_1, g_2 be a pair of hyperbolic metrics on M. We say a map $u : (M, g_1) \to (M, g_2)$ is a minimal map if the graph of u is a minimal surface in $M \times M$.

Theorem 0.1. *There is a unique, area preserving, minimal map $u : (M, g_1) \to (M, g_2)$ homotopic to the identity.*

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The theorem has since been generalized by Y-I. Lee [**?**].

In this result the surface M is compact so boundary considerations do not arise. However motivated by the theorem we consider the following:

Problem 1. *Let D_1 and D_2 be connected, simply connected, closed, bounded domains in $\mathbb{R}^2$ with smooth boundaries and with equal areas. Find an area preserving diffeomorphism $\psi : D_1 \to D_2$ smooth up to the boundary such that the graph of ψ, graph(ψ), is a minimal surface in $\mathbb{R}^4 \simeq \mathbb{R}^2 \times \mathbb{R}^2$.*

Consider the symplectic form $\omega_1 = dx_1 \wedge dy_1 - dx_2 \wedge dy_2$ on $\mathbb{R}^4 \simeq \mathbb{R}^2 \times \mathbb{R}^2$, where $dx_i \wedge dy_i$ are the standard area forms on $D_i \subset \mathbb{R}^2$, $i = 1, 2$. A diffeomorphism $\psi : D_1 \to D_2$ is area preserving if and only if its graph is a lagrangian surface in $(\mathbb{R}^4, \omega_1)$. Hence the problem can be reformulated as: *Find a diffeomorphism $\psi : D_1 \to D_2$ smooth up to the boundary such that graph (ψ) is a minimal lagrangian surface in* $(\mathbb{R}^4, \omega_1)$. We will call such a diffeomorphism a *minimal lagrangian diffeomorphism.* A minimal lagrangian diffeomorphism $\psi : D_1 \to D_2$ determines a minimal lagrangian surface, graph ψ, with boundary lying on the lagrangian torus $T^2 = \partial D_1 \times \partial D_2$. We are thus led to consider a free boundary problem for minimal lagrangian surfaces.

The subject of minimal lagrangian surfaces is relatively new with only rather preliminary results. We have devoted §1 to a discussion of some of these results attempting to unify the various points of view around the ideas of the *lagrangian angle* and the *Maslov form.*

In §2 we use the lagrangian angle to show that there are pairs of domains D_1 and D_2 for which there is no minimal lagrangian diffeomorphism $D_1 \to D_2$. Given a pair of domains D_1 and D_2 consider the lagrangian torus $T^2 = \partial D_1 \times \partial D_2 \subset \mathbb{R}^4$. In §1 we show that on T^2 there is a function β_{T^2}, the lagrangian angle, well defined mod $2\mathbb{Z}$. Let $\phi : D_1 \to D_2$ be an orientation preserving diffeomorphism. The boundary trace of ϕ determines a $(1, 1)$ curve, Γ, on T^2 along which the lagrangian angle, β_{T^2}, is a well defined function. We define,

$$variation(\phi) = sup_{x,y \in \Gamma} |\beta_{T^2}(x) - \beta_{T^2}(y)|.$$

We define the $variation(D_1, D_2)$ to be the infimum of $variation(\phi)$ over all diffeomorphisms $\phi : D_1 \to D_2$. Then,

Theorem 2.6. *Let D_1 and D_2 be connected, simply connected, closed, bounded domains in $\mathbb{R}^2$ with smooth boundary and with equal* *areas. Suppose that,*

$$variation(D_1, D_2) \geq 1.$$

Then there are no minimal lagrangian diffeomorphisms $\psi : D_1 \to D_2$ smooth up to the boundary.

It is easy to find pairs of domains (D_1, D_2) satisfying

$$variation(D_1, D_2) \geq 1.$$

Roughly speaking, $variation(D_1, D_2)$ measures the difference of the curvatures of the boundary curves $\partial D_i \subset \mathbb{R}^2$. On the other hand, if both domains D_i are convex, then $variation(D_1, D_2) < 1$.

In §3 we prove an implicit function theorem which implies that if a pair (D_1, D_2) of domains admits a minimal lagrangian diffeomorphism $\psi : D_1 \to D_2$, then any pair $(\tilde{D}_1, \tilde{D}_2)$ sufficiently close to (D_1, D_2) also admits a minimal lagrangian diffeomorphism $\tilde{\psi} : \tilde{D}_1 \to \tilde{D}_2$. This result is based on a study of a Riemann-Hilbert boundary system that arises from the linearization of the equations determining a minimal lagrangian diffeomorphism.

§4 and §5 are devoted to proving an existence theorem for minimal lagrangian diffeomorphisms. To describe the result let κ_i denote the curvature of ∂D_i in $\mathbb{R}^2$. Suppose that D_1 and D_2 are connected, simply connected, closed, bounded domains. We say the pair (D_1, D_2) is *pseudoconvex* if,

$$\min_{\partial D_1} \kappa_1 + \min_{\partial D_2} \kappa_2 > 0.$$

Note that if (D_1, D_2) is pseudoconvex, one of the domains may not be convex. We prove,

Theorem 5.1. *Let (D_1, D_2) be a pseudoconvex pair of domains with smooth boundaries, satisfying area(D_1) = area(D_2). Then there is a minimal lagrangian diffeomorphism $\psi : D_1 \to D_2$, smooth up to the boundary.*

The proof of this theorem uses convergence properties of $J-$holomorphic discs similar in spirit to arguments of Bedford-Gaveau [**?**] and Gromov [**?**]. However, unlike the arguments of [**?**] and [**?**], in our setting the boundaries of the holomorphic discs lie on a surface that contains complex tangent points. The pseudoconvexity condition on the pair (D_1, D_2) insures that the boundaries of the holomorphic discs are bounded away from the complex tangent points and hence the discs can be shown to converge.

The problem from analysis is more classical.

Problem 2. *Let D_1 and D_2 be connected, simply connected, closed, bounded domains in $\mathbb{R}^2$ with smooth boundaries and with equal areas. Find a smooth function w on D_1 satisfying the Monge-Ampère equation:*

$$w_{xx}w_{yy} - (w_{xy})^2 = 1,$$

such that the gradient of w, ∇w, defines a diffeomorphism $D_1 \to D_2$.

Problem 2 is a boundary value problem for the Monge-Ampère equation. Following the terminology of Pogorelov [**?**], it is known as the "second boundary value problem." (The "first boundary value problem" is the Dirichlet problem.) In the 1950's assuming both domains are convex Pogorelov produced a "generalized" solution in the sense of A. D. Alexandrov. More recently, Brenier [**?**] showed the existence and uniqueness of a weak solution for domains in any dimension such that the Lebesgue measure of their boundaries is zero. Thus the problem is a question of the regularity of the solution. Assuming both domains are strictly convex and two dimensional Delanoë [**?**] in 1989 proved regularity. In 1991 Caffarelli in a series of papers (see in particular [**?**] and [**?**]) proved regularity for convex domains in arbitrary dimensions. Caffarelli [**?**] also gave an example to show that if convexity is not assumed regularity can be false. We remark that the work of Brenier, Delanoë and Caffarelli allows more general functions on the right-hand side of the above equation than considered here.

We observe in §4 that Problems 1 and 2 are essentially equivalent. In fact, the gradient of a solution of Problem 2 is a solution of Problem 1, and a solution of Problem 1 determines a solution of Problem 2. It follows that the existence and non-existence results that we derived for minimal lagrangian diffeomorphisms imply analogous results for the second boundary value problem for the Monge-Ampère equation. In particular, the obstruction to existence we describe in §2 gives a geometric necessary condition on pairs of domains (D_1, D_2) for a solution of the second boundary value problem:

Theorem 4.1. *Let (D_1, D_2) be a pair of connected, simply connected, closed, bounded domains in $\mathbb{R}^2$ with smooth boundaries and equal areas. If*

$$variation(D_1, D_2) \geq 1,$$

then there is no regular solution of the second boundary-value problem for the Monge-Ampère equation.

The existence result of §5 gives an existence theorem for the second boundary value problem that includes and extends the regularity theorem of Delanoë.

Corollary 5.2. *Let (D_1, D_2) be a pseudoconvex pair of domains with smooth boundaries, satisfying area$(D_1) =$ area(D_2). Then there is a solution of the second boundary value problem for the Monge-Ampère equation smooth up to the boundary, that is, there is a smooth function w on D_1 satisfying,*

$$w_{xx}w_{yy} - (w_{xy})^2 = 1$$

such that the gradient of w, ∇w, defines a diffeomorphism $(D_1, \partial D_1) \to (D_2, \partial D_2)$.

Theorem 4.1, Corollary 5.2 and their proofs show that convexity is not central to the existence and nonexistence of smooth solutions of the second boundary value problem. Rather, more subtle geometry of the torus $T^2 = \partial D_1 \times \partial D_2$, such as the location on T^2 of J-complex tangent points, plays a more fundamental role. However, the general problem of giving necessary and sufficient conditions for the solution of Problems 1 and 2 remains open.

Throughout this paper all domains D in $\mathbb{R}^2$ will be assumed to have *smooth boundary* and to be *connected*, *simply connected*, *closed* and *bounded* unless otherwise noted.

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1. Lagrangian submanifolds and the Maslov form

We will be considering the graphs of area-preserving maps $D_1 \to D_2$, or equivalently, lagrangian surfaces in $\mathbb{R}^4$. The purpose of this section is to develop the requisite geometry. It is certainly possible to do this simply for lagrangian surfaces in $\mathbb{R}^4$. But, as we will see, it is not more difficult to describe this geometry in the more general setting of lagrangian immersions in Kähler manifolds. Moreover, in the general setting, the relation between the topology and geometry of the lagrangian immersion and that of the ambient manifold becomes clear.

Let X be a Kähler manifold of complex dimension n, with Kähler form ω and complex structure J. Let L be a smooth connected oriented manifold of real dimension n, and let $\ell : L \to X$ be a lagrangian immersion. Let $\{\theta_1, \dots, \theta_n, \eta_1, \dots, \eta_n\}$ be an orthonormal coframe adapted to L it follows that:

(i) $\{\theta_1 \dots \theta_n\}$ is an orthonormal coframe on L for the induced metric.

(ii) $\eta_1 = \cdots = \eta_n = 0$ on L.

(iii) $J\eta_j = \theta_j, \quad j = 1, \dots, n,$

The 1-forms

$$\omega_j = \theta_j + i\ \eta_j, \quad j = 1, \dots, n, \tag{1.1}$$

form a unitary coframe of $\ell^* TX$ adapted to L. Let $(\omega_{j\bar{k}})$ denote the connection 1-forms with respect to this coframe. Set,

$$\tau = \frac{1}{\pi}(i \sum_k \omega_{k\bar{k}}). \tag{1.2}$$

An easy computation yields,

Proposition 1.1. *τ is a well-defined real valued 1-form on L.*

Denote the curvature two-form of X by $(\Omega_{j\bar{k}})$. Then

$$d(i \sum_k \omega_{k\bar{k}}) = i \sum_k \Omega_{k\bar{k}} = \text{Ric}, \tag{1.3}$$

Suppose that X is Kähler-Einstein ($\text{Ric} = \text{R}\ \omega$) and $\ell : L \to X$ is a lagrangian immersion. From (**??**) it follows that τ is closed. Hence τ represents a cohomology class $[\tau] \in H^1(L; \mathbb{R})$.

Definition 1.1. The closed one-form τ is called the *Maslov form.*

Let K and ∇ denote the canonical line bundle on X and its induced connection, respectively. The curvature of $\ell^*\nabla$ satisfies:

$$c_1(\ell^*\nabla) = \ell^*\text{Ric} = \text{R}\ell^*\omega = 0, \tag{1.4}$$

since ℓ is lagrangian. Thus $\ell^*\nabla$ is a flat connection on $\ell^* K$. The holonomy of $\ell^*\nabla$ is, then, an element of $\text{Hom}(H_1(L;\mathbb{Z}), S^1) \simeq H^1(L; S^1)$ which we denote $\text{Hol}(\ell)$.

The holonomy and the Maslov form τ are closely related. To see this consider the short exact sequence

$$0 \to 2\mathbb{Z} \to \mathbb{R} \xrightarrow{e} S^1 \to 0 \tag{1.5}$$

where $e(t) = e^{\pi i t}$. This determines the long exact cohomology sequence

$$\begin{array}{ccccccc} 0 \to H^1(L;\mathbb{Z}) \xrightarrow{i} H^1(L;\mathbb{R}) \xrightarrow{e} H^1(L;S^1) \xrightarrow{\delta} & H^2(L;\mathbb{Z}) & \to H^2(L;\mathbb{R}) \\ & \uparrow \; c_1 & \\ & H^1(L;\mathcal{C}^*) & \end{array}$$

where $H^1(L;\mathcal{C}^*)$ is the space of C^∞ complex line bundles over L. Applying the sequence to the line bundle ℓ^*K, we obtain that $\delta(\mathrm{Hol}(\ell)) = 0$, since $c_1(\ell^*K) = 0$. From (1.3) it follows that the cohomology class $[\tau] \in H^1(L;\mathbb{R})$ satisfies $e([\tau]) = \mathrm{Hol}(\ell)$. Thus if $\mathrm{Hol}(\ell) = 0$ (i.e., the connection $\ell^*\nabla$ has no holonomy), then $[\tau]$ is an element of $H^1(L;\mathbb{Z})$.

Definition 1.2. When $\mathrm{Hol}(\ell) = 0$ the cohomology class $[\tau] \in H^1(L;\mathbb{Z})$ is called the *Maslov class* of ℓ and denoted $\mathrm{Mas}(\ell)$.

The classical definitions of the Maslov class assume $\pi_1(X) = 0$ (and hence $\mathrm{Hol}(\ell) = 0$). Our definition reduces to the classical one when $\pi_1(X) = 0$.

To give a description of the Maslov class of a lagrangian immersion ℓ which is more suitable for computation, let $\pi : \ell^*K \to L$ denote the bundle projection and suppose that $\mathrm{Hol}(\ell) = 0$. Choose $x_0 \in L$ and $v \in \pi^{-1}(x_0)$ with $|v| = 1$. Use the flat hermitian connection $\ell^*\nabla$ to construct, by parallel translation, a parallel section σ of ℓ^*K with $|\sigma| = 1$ and $\sigma(x_0) = v$. For each point $x \in L$, $\sigma(x)$ is a unit $(n,0)$ form at $\ell(x) \in X$. Let T_x denote the oriented unit tangent n-plane of L at x considered as a subspace of $T_{\ell(x)}X$. We define a function β on L with values in $\mathbb{R}/2\mathbb{Z}$ by,

$$\sigma(x)(T_x) = e^{\pi i \beta(x)}. \tag{1.6}$$

Definition 1.3. The function β is called the *lagrangian angle* of ℓ.

β depends on the choice of $v \in \pi^{-1}(x_0)$, $|v| = 1$. A different choice will change β by the addition of a constant. Hence, $d\beta$ is a well-defined closed 1-form on L. $d\beta$ represents a cohomology class in $H^1(L;\mathbb{Z})$.

Theorem 1.2. *$d\beta = \tau$ and therefore $[d\beta] = Mas(\ell)$.*

Proof. Let σ be a parallel section of ℓ^*K of unit length. Choose a unitary coframe $\{\omega_1, \ldots, \omega_n\}$ adapted to L as described above. Then,

$$\sigma = e^{\pi i \beta}\theta_1 \wedge \cdots \wedge \theta_n = e^{\pi i \beta}\omega_1 \wedge \cdots \wedge \omega_n, \tag{1.7}$$

since $\omega_j = \theta_j$, $j = 1, \ldots, n$, along L. Let ∇ denote the connection $\ell^*\nabla$ on ℓ^*K. Then

$$0 = \nabla\sigma = (\pi i d\beta + \sum_j \omega_{j\bar{j}}) \otimes \sigma. \tag{1.8}$$

Hence, $\pi d\beta = i\sum_j \omega_{j\bar{j}} = \pi\ \tau$. q.e.d.

Remark 1.1. When $K \to X$ is trivial and the compatible connection ∇ has no holonomy, a simpler definition of the lagrangian angle is possible. Let σ be a parallel section of $K \to X$ such that $|\sigma| = 1$. For a lagrangian immersion ℓ the n-form $\ell^*\sigma$ has unit length. Hence, we can write

$$(\ell^*\sigma)(x) = e^{\pi i\beta(x)} dvol_x, \tag{1.9}$$

where $dvol_x$ is the volume form on L determined by the Riemannian metric induced by ℓ. More generally, we can define a lagrangian angle $\beta_{\mathcal{P}}$ on the Grassmann bundle $\mathcal{P} \to X$ of oriented lagrangian n-planes in TX, as follows: For each $x \in X$ and each unit lagrangian n-plane P_x in T_xX, set

$$\sigma(x)(P_x) = e^{\pi i\beta_{\mathcal{P}}(P_x)}. \tag{1.10}$$

Then $\beta_{\mathcal{P}}$ is a function on $\mathcal{P}$ with values in $\mathbb{R}/2\mathbb{Z}$.

Remark 1.2. The above treatment can be formulated for L unoriented. In this case the lagrangian angle, defined as above, is well-defined mod $\mathbb{Z}$.

Next we relate the lagrangian angle and the Maslov form to more classical geometric invariants.

Theorem 1.3. *Suppose X is a Kähler-Einstein manifold and $\ell : L \to X$ is a lagrangian immersion. Let H denote the mean curvature vector field of L in X. Then*

$$\tau = \frac{1}{\pi}(H \lrcorner\, \omega).$$

In particular, the one-form $\frac{1}{\pi}(H \lrcorner\, \omega)$ on L is closed. When L has no holonomy (i.e.,the line bundle ℓ^*K has no holonomy), then $\frac{1}{\pi}(H \lrcorner\, \omega)$ represents the Maslov class of ℓ in $H^1(L;\mathbb{Z})$.

Proof. Left to the reader q.e.d.

Corollary 1.4. *If X is a Kähler-Einstein manifold and $\ell : L \to N$ is a lagrangian immersion, then the mean curvature vector H is an infinitesimal symplectic motion. Equivalently, H is tangent to the space of lagrangian submanifolds near L.*

Proof. The Lie derivative of ω in the direction H is given by,

$$\mathcal{L}_H(\omega) = d(H \lrcorner \, \omega) + H \lrcorner \, d\omega.$$

The result follows. q.e.d.

Remark 1.3. When $X = \mathbb{C}^n$ with its standard hermitian metric, the theorem and its corollary occur in Harvey-Lawson [**?**]. When X is Ricci-flat and simply-connected they occur in Dazard [**?**]. When $X = \mathbb{C}P^2$ with the Fubini-Study metric they occur in [**?**], and as described here the results are due to Bryant [**?**].

Further we have,

Corollary 1.5. *Suppose X is a Kähler-Einstein manifold and ℓ is a lagrangian immersion.*

(i) If ℓ is a minimal immersion, then $Hol(\ell) = 0$, $Mas(\ell) = 0$ and β is constant on each component of L.

(ii) If $Mas(\ell) = 0$, then β is a well-defined function on L with values in $\mathbb{R}$.

The condition $\mathrm{Mas}(\ell) = 0$ implicitly assumes that $\mathrm{Hol}(\ell) = 0$.

When X has complex dimension 2, the lagrangian angle β has some special properties. First, suppose $X = \mathbb{C}^2 \simeq \mathbb{R}^4$ with the standard Kähler structure. We have already observed in Remark (1.1) that in this case the lagrangian angle β can be defined as a function with values in $\mathbb{R}/2\mathbb{Z}$ on the space of oriented lagrangian 2-planes in $\mathbb{R}^4$.

Proposition 1.6. *If P_i , $i = 1, 2$, are oriented lagrangian 2-planes in $\mathbb{R}^4$ satisfying $\beta(P_1) \equiv \beta(P_2)$ mod $\mathbb{Z}$, then there is an orthogonal complex structure J on $\mathbb{R}^4$ such that either P_1 and P_2 are J complex lines or P_1 and $-P_2$ are J complex lines, where $-P_2$ denotes the 2-plane P_2 with the orientation reversed.*

Proof. Observe that if P_1 and P_2 are oriented lagrangian 2-planes, then $\beta(P_1) \equiv \beta(P_2) \mod \mathbb{Z}$ implies either $\beta(P_1) \equiv \beta(P_2) \mod 2\mathbb{Z}$ or $\beta(P_1) \equiv \beta(P_2) + 1 \mod 2\mathbb{Z}$. We begin by supposing that $\beta(P_1) \equiv \beta(P_2) \mod 2\mathbb{Z}$. Without loss of generality we can suppose that P_1 is the lagrangian plane $\{y_1 = y_2 = 0\}$ and so $\beta(P_1) \equiv 0 \mod 2\mathbb{Z}$. First suppose that P_1 and P_2 intersect in a line. Changing coordinates, if necessary, we can suppose that the line of intersection is $\{y_1 = y_2 = x_2 = 0\}$. Using the holomorphic $(2,0)$-form $dz_1 \wedge dz_2$ on $\mathbb{C}^2$ and (**??**) it follows that $\beta(P_2) \not\equiv 0 \mod 2\mathbb{Z}$. Hence we can suppose that P_1 and P_2 intersect only in the origin. Thus P_2 is defined by the equations:

$$\begin{aligned} y_1 &= a_1x_1 + a_2x_2 \\ y_2 &= a_2x_1 + a_3x_2. \end{aligned}$$

Using the holomorphic $(2,0)$-form $dz_1 \wedge dz_2$ and $\beta(P_2) \equiv 0 \mod 2\mathbb{Z}$ we obtain that, $a_1 + a_3 = 0$. Define the orthogonal complex structure J by,

$$J : \tfrac{\partial}{\partial x_1} \mapsto \tfrac{\partial}{\partial x_2}, \;\; \tfrac{\partial}{\partial x_2} \mapsto -\tfrac{\partial}{\partial x_1}; \;\; \tfrac{\partial}{\partial y_1} \mapsto -\tfrac{\partial}{\partial y_2}, \;\; \tfrac{\partial}{\partial y_2} \mapsto \tfrac{\partial}{\partial y_1}.$$

Clearly both P_1 and P_2 are J-complex.

Now suppose that $\beta(P_1) \equiv \beta(P_2) + 1 \mod 2\mathbb{Z}$. From the equality $\beta(-P_2) = \beta(P_2) + 1$ and the above argument it follows that both P_1 and $-P_2$ are J-complex. q.e.d.

Remark 1.4. Suppose $X \simeq \mathbb{C} \times \overline{\mathbb{C}}$, that is, suppose $X = \mathbb{C}^2$ with the Kähler form, $dz_1 \wedge d\overline{z}_1 + d\overline{z}_2 \wedge dz_2$. Proposition **??** remains true.

2. An obstruction to existence

We begin this section by computing the lagrangian angle and the Maslov form in the simplest situation – that of a simple closed curve in $\mathbb{R}^2 \simeq \mathbb{C}$.

Suppose that C is a simple closed curve in $\mathbb{C}$ parameterized by,

$$c : [0,1] = I \to \mathbb{R}^2 \tag{2.1}$$

with $c(0) = c(1)$. Let $\{e, f\}$ be the Frenet frame along c. That is, $\{e, f\}$ is an oriented orthonormal frame along $c(t)$ satisfying:

$$\begin{aligned} c'(t) &= |c'(t)|e(t) \\ e'(t) &= \kappa(t)|c'(t)|f(t) \end{aligned} \tag{2.2}$$

Choose a unit vector $v \in \mathbb{R}^2$ and define the angle $\theta(t)$ by,

$$\begin{aligned} \cos \pi\theta(t) &= e(t) \cdot v, \\ \sin \pi\theta(t) &= -\, f(t) \cdot v. \end{aligned} \tag{2.3}$$

$\theta(t)$ is well defined mod $2\mathbb{Z}$ and depends on the choice of the vector v. However, $\theta'(t)$ is well-defined independent of all choices. It is well known that $\theta(t)$ is a primitive of the curvature in the sense that,

$$\pi\theta'(t) = \kappa(t)|c'(t)|. \tag{2.4}$$

The choice of v is equivalent to the choice of a parallel $(1,0)$ form, dz, of unit length as follows: Given v choose euclidean coordinates (x, y) such that,

$$v = \tfrac{\partial}{\partial x}, \quad Jv = \tfrac{\partial}{\partial y},$$

and let $dz = dx + idy$. By (2.3) we have,

$$dz(e) = \exp(i\pi\theta). \tag{2.5}$$

Since $e(t)$ is the unit tangent space of C at $c(t)$, it follows from (**??**) that $\theta(t)$ is the lagrangian angle along C. Thus we have shown,

Proposition 2.1. *Let C be a simple closed curve in $\mathbb{R}^2 \simeq \mathbb{C}$ parameterized by $c : I \to \mathbb{C}$. Let $\kappa(t)$ denote the curvature function of c and let $\theta(t)$ be a primitive of the curvature. Then:*

(i) θ is the lagrangian angle on C,

(ii) the Maslov class $Mas(c)$ is represented by the Maslov one-form,

$$d\theta = \tfrac{1}{\pi}\kappa(t)|c'(t)|dt.$$

We next consider the computation of the lagrangian angle and the Maslov class on product tori in $\mathbb{R}^2 \times \mathbb{R}^2$. Denote the projections onto the first and second factors of $\mathbb{R}^2 \times \mathbb{R}^2$ by π_1 and π_2, respectively. Consider simple closed curves $C_j \subset \mathbb{R}^2 \simeq \mathbb{C}$, $j = 1, 2$. Suppose that C_j is parameterized by $c_j : I \to \mathbb{C}$. Let κ_j denote the curvature of c_j and let θ_j denote a primitive of κ_j. Let (x_j, y_j) be euclidean coordinates on $\mathbb{R}^2$. Then the symplectic forms ω_+ and ω_- on $\mathbb{R}^4 \simeq \mathbb{R}^2 \times \mathbb{R}^2$ are:

$$\begin{aligned} \omega_+ &= dx_1 \wedge dy_1 + dx_2 \wedge dy_2, \\ \omega_- &= dx_1 \wedge dy_1 - dx_2 \wedge dy_2. \end{aligned} \tag{2.6}$$

The product $C_1 \times C_2 \subset \mathbb{R}^4$ is a lagrangian torus without holonomy for both symplectic forms.

Proposition 2.2. *On the lagrangian torus*

$$T^2 \simeq C_1 \times C_2 \subset (\mathbb{R}^2 \times \mathbb{R}^2, \omega_\pm)$$

the lagrangian angle is

$$\beta_\pm = {\pi_1}^*\theta_1 \pm {\pi_2}^*\theta_2. \tag{2.7}$$

The Maslov form is

$$\tau_\pm = \tfrac{1}{\pi}({\pi_1}^*(\kappa_1(t_1)|c_1'(t_1)|dt_1) \pm {\pi_2}^*(\kappa_2(t_2)|c_2'(t_2)|dt_2)). \tag{2.8}$$

Proof. Clear. q.e.d.

For the remainder of the paper we will restrict our attention to $\mathbb{R}^4 \simeq \mathbb{C}\times\overline{\mathbb{C}}$. That is, we consider $\mathbb{C}^2 \simeq \mathbb{C}\times\overline{\mathbb{C}}$ equipped with the Kähler structure determined by the Kähler form $\frac{i}{2}(dz_1 \wedge d\overline{z}_1 + d\overline{z}_2 \wedge dz_2)$ and the euclidean metric. If

$$z_1 = x_1 + iy_1, \quad z_2 = x_2 + iy_2,$$

the symplectic form on $\mathbb{C} \times \overline{\mathbb{C}}$ is

$$dx_1 \wedge dy_1 - dx_2 \wedge dy_2. \tag{2.9}$$

We will henceforth denote this form by ω.

Let D_1 and D_2 be domains in $\mathbb{R}^2$ with smooth boundaries, and let $\psi : D_1 \to D_2$ be an area-preserving diffeomorphism smooth up to the boundary. Since ψ is area-preserving, its graph, $graph(\psi)$, is a lagrangian surface in $(\mathbb{R}^4, \omega)$. Denote by β_ψ the lagrangian angle along $graph(\psi)$. Consider the lagrangian torus $T^2 = \partial D_1 \times \partial D_2$, as above. Let β_{T^2} denote the lagrangian angle along T^2. We have:

Theorem 2.3. *Let D_1 and D_2 be domains in $\mathbb{R}^2$ with smooth boundaries and $\psi : D_1 \to D_2$ an area-preserving diffeomorphism smooth up to the boundary. Then at each point $(p, \psi(p))$, $p \in \partial D_1$, of the boundary of $graph(\psi)$:*

$$\beta_\psi(p, \psi(p)) \neq \beta_{T^2}(p, \psi(p)) \ \text{ in } \mathbb{R}/\mathbb{Z}. \tag{2.10}$$

Proof. Suppose, to the contrary, that there is a point $(p, \psi(p))$, $p \in \partial D_1$ such that $\beta_\psi(p, \psi(p)) \equiv \beta_{T^2}(p, \psi(p)) \bmod \mathbb{Z}$. By Proposition **??** and Remark 1.4 this implies that there is an orthogonal complex structure J on $\mathbb{R}^4$ such that, the (unoriented) tangent planes of $graph(\psi)$ and

T^2 at $(p, \psi(p))$ are J-complex lines. But these $2-$planes intersect in a real line. Since they are J-complex they must coincide. Thus the graph of ψ is tangent to T^2 at $(p, \psi(p))$ and so ψ cannot be a diffeomorphism at p. This contradiction establishes the theorem. q.e.d.

Let $\psi : D_1 \to D_2$ be any orientation-preserving diffeomorphism smooth up to the boundary. The boundary trace of ψ determines a smooth $(1,1)$ curve, γ, on T^2 satisfying $\langle \mathrm{Mas}(T^2), \gamma \rangle = 0$. It follows from Corollary 1.5(ii) that along γ the lagrangian angle β_{T^2} is a smooth function with values in $\mathbb{R}$, well defined up to normalization.

Definition 2.1. The *variation* of ψ, denoted $variation(\psi)$, is:

$$variation(\psi) = \sup_{x,y \in \gamma} |\beta_{T^2}(x) - \beta_{T^2}(y)|. \tag{2.11}$$

Note that the difference $\beta_{T^2}(x) - \beta_{T^2}(y)$, for $x, y \in \gamma$, is well defined independent of choice of normalization.

Let $\mathcal{A}(D_1, D_2)$ denote the set of area-preserving diffeomorphisms $D_1 \to D_2$ smooth up to the boundary.

Definition 2.2. The $variation_{\mathcal{A}}$ of the pair (D_1, D_2), denoted $variation_{\mathcal{A}}(D_1, D_2)$, is:

$$variation_{\mathcal{A}}(D_1, D_2) = inf_{\psi \in \mathcal{A}(D_1,D_2)} variation(\psi). \tag{2.12}$$

Let $\mathcal{D}(D_1, D_2)$ denote the set of orientation preserving diffeomorphisms $D_1 \to D_2$ smooth up to the boundary.

Definition 2.3. The $variation_{\mathcal{D}}$ of the pair (D_1, D_2), denoted $variation_{\mathcal{D}}(D_1, D_2)$, is:

$$variation_{\mathcal{D}}(D_1, D_2) = inf_{\psi \in \mathcal{D}(D_1,D_2)} variation(\psi). \tag{2.13}$$

Lemma 2.4. *If D_1 and D_2 are domains of equal area with smooth boundaries then,*

$$variation_{\mathcal{A}}(D_1, D_2) = variation_{\mathcal{D}}(D_1, D_2).$$

Proof. Clearly,

$$variation_{\mathcal{A}}(D_1, D_2) \geq variation_{\mathcal{D}}(D_1, D_2).$$

On the other hand, by Theorem 1 of Dacorogna-Moser [**?**], given any $\psi_0 \in \mathcal{D}(D_1, D_2)$ there is an area-preserving diffeomorphism $\psi \in \mathcal{A}(D_1, D_2)$ with $\psi = \psi_0$ on ∂D_1. q.e.d.

Because of the lemma we can denote both $variation_{\mathcal{A}}$ and $variation_{\mathcal{D}}$ by *variation*.

Theorem 2.5. *Let D_1 and D_2 be domains in $\mathbb{R}^2$ with smooth boundaries and equal areas. Suppose that,*

$$variation(D_1, D_2) \geq 1.$$

Then there are no minimal lagrangian diffeomorphisms $\psi : D_1 \to D_2$ smooth up to the boundary.

Proof. Suppose such a diffeomorphism $\psi : D_1 \to D_2$ exists. Then $graph(\psi)$ is a minimal lagrangian surface in $(\mathbb{R}^4, \omega)$ and so β_ψ is constant on $graph(\psi)$. On the other hand since $variation(D_1, D_2) \geq 1$, it follows that $variation(\psi) \geq 1$. Thus, along the boundary of $graph(\psi)$, β_{T^2} assumes every value in $\mathbb{R}/\mathbb{Z}$. Therefore there is at least one point $(p, \psi(p))$, $p \in \partial D_1$, such that,

$$\beta_{T^2}(p, \psi(p)) = \beta_\psi \quad \text{in } \mathbb{R}/\mathbb{Z}.$$

The result now follows from Theorem 2.3. q.e.d.

Remark 2.1. We included Proposition 2.2 in this section because it shows that the computation of $variation(D_1, D_2)$ reduces to comparing the primitive of the curvature of ∂D_1 to the primitive of the curvature of ∂D_2. Consequently, it is easy to construct pairs of domains (D_1, D_2) with equal areas and with $variation(D_1, D_2) \geq 1$. On the other hand note that $variation(D, D) = 0$ for any domain D. Thus,if ∂D_2 is close to ∂D_1 in C^1 then, by continuity, $variation(D_1, D_2) < 1$. Also, again using Proposition 2.2, it follows that if both D_1 and D_2 are convex, then $variation(D_1, D_2) < 1$.

For use later in the paper we record:

Theorem 2.6. *Let C_j, $j = 1, 2$, be simple closed curves in $\mathbb{R}^2$ with curvature functions κ_j, $j = 1, 2$. Let $\phi : C_1 \to C_2$ be a diffeomorphism. Suppose that C_j, $j = 1$ or $j = 2$, is strictly convex (i.e., one of $\kappa_1 > 0$ or $\kappa_2 > 0$). Then,*

$$length(graph(\phi)) < B,$$

where B depends on the Maslov class of $T^2 = C_1 \times C_2$ and on the curvatures κ_j, $j = 1, 2$, but is independent of ϕ.

Proof. Suppose first that C_2 is strictly convex. The Maslov class, $\mathrm{Mas}(T^2)$, of $T^2 = C_1 \times C_2$ pairs with any class $\alpha \in H_1(T^2; \mathbb{Z})$ to determine an integer $\langle \mathrm{Mas}(T^2), \alpha \rangle$. In particular, if γ is the $(1, 1)$ class in

$H_1(T^2;\mathbb{Z})$, then γ can be represented by the graph of ϕ. Hence, we can compute $\langle \mathrm{Mas}(T^2),\gamma\rangle$ by

$$\langle \mathrm{Mas}(T^2),\gamma\rangle = \int_{\text{graph }\phi} \tau_-, \tag{2.14}$$

where τ_- is the Maslov form given by (**??**). Let $c_1 : I \to \mathbb{R}^2$ be the parameterization of C_1 by arclength. The curve determined by graph ϕ can be parameterized by,

$$\begin{aligned} c : I &\to \mathbb{R}^4 \\ t &\mapsto (c_1(t), \phi(c_1(t))). \end{aligned} \tag{2.15}$$

Thus,

$$\begin{aligned} \int_{\text{graph }\phi} \tau_- &= \int_{c(I)} \tau_- \\ &= \tfrac{1}{\pi}\int_I \kappa_1(t)dt - \tfrac{1}{\pi}\int_I \kappa_2(t)|(\phi\circ c_1)'(t)|dt \\ &= 2 - \tfrac{1}{\pi}\int_I \kappa_2(t)|\phi'(c_1(t))||c_1'(t)|dt, \end{aligned} \tag{2.16}$$

where the first term of the last line is due to the "Umlaufsatz". Hence, since $\kappa_2 > 0$, (**??**) and (2.16) imply

$$\int_I |\phi'(c_1(t))||c_1'(t)|dt < A, \tag{2.17}$$

where A is independent of ϕ. It follows then from (2.17) that length $(c) < B$, as required.

If, on the other hand, C_1 is strictly convex, then apply the above argument to $\phi^{-1} : C_2 \to C_1$. Since graph $(\phi) =$ graph (ϕ^{-1}) the result follows. q.e.d.

3. Minimal lagrangian diffeomorphisms: local theory

Let $D_i,\ i = 1,2$, be domains in $\mathbb{R}^2$ with smooth boundary. Let $r_i,\ i = 1,2$, be C^∞ defining functions $\mathbb{R}^2 \to \mathbb{R}$ such that:

(i) $D_i = \{(s,t) \in \mathbb{R}^2 : r_i(s,t) \le 0\}$,

(ii) $(\text{grad } r_i)|_{\partial D_i} \ne 0$.

Suppose that $\psi : D_1 \to D_2$ is a minimal lagrangian diffeomorphism smooth up to the boundary. Let (x, y) be euclidean coordinates on D_1 and (u, v) be euclidean coordinates on D_2. Then

$$\begin{aligned} \psi(x,y) =& (u(x,y), v(x,y)), \\ r_2(u(x,y), v(x,y)) =& 0 \text{ whenever } r_1(x,y) = 0. \end{aligned} \tag{3.1}$$

Since ψ is area-preserving, we have

$$u_x v_y - u_y v_x = 1. \tag{3.2}$$

Since the graph of ψ is a minimal surface, the surface

$$(x,y) \overset{f_\psi}{\mapsto} (x, y, u(x,y), v(x,y)) \tag{3.3}$$

is a minimal lagrangian surface in $(\mathbb{R}^4, dx \wedge dy - du \wedge dv)$. Hence the lagrangian angle β is constant along (**??**).

We compute β as follows: Let $z_1 = x + iy$ and $z_2 = u + iv$. Then $dz_1 \wedge d\bar{z}_2$ is a parallel section of the canonical line bundle K over $\mathbb{R}^4 \simeq \mathbb{C} \times \bar{\mathbb{C}}$. Thus,

$$\begin{aligned} {f_\psi}^*(dz_1 \wedge d\bar{z}_2) =& (dx + idy) \wedge (du - idv) \\ =& [(u_y - v_x) - i(u_x + v_y)] dx \wedge dy. \end{aligned} \tag{3.4}$$

From (**??**) we have,

$$\frac{-(u_x + v_y)}{u_y - v_x} = \tan(\pi\beta). \tag{3.5}$$

Therefore, β is constant along (**??**) if and only if

$$u_y - v_x = \gamma(u_x + v_y), \tag{3.6}$$

where $\gamma = -\cot(\pi\beta)$ is a constant.

Proposition 3.1. *The map $\psi = (u, v) : D_1 \to D_2$ is a minimal lagrangian diffeomorphism if and only if on D_1:*

$$\begin{aligned} u_x v_y - u_y v_x &= 1, \\ u_y - v_x &= \gamma(u_x + v_y), \\ r_2(u,v) = 0 \quad &\mathit{if} \quad r_1(x,y) = 0 \end{aligned} \tag{3.7}$$

for some constant γ.

Proof. We have already shown that if ψ is a minimal lagrangian diffeomorphism, then (3.7) holds. Conversely, the first equation of (3.7) shows that ψ is a diffeomorphism and is area-preserving. The second equation shows that ψ has a minimal graph. q.e.d.

Consider a family $D_2(t)$, $t \in (-\delta\ ,\delta\ )$ $\delta > 0$, of domains in $\mathbb{R}^2$ with smooth boundary. Suppose that:

(i) $D_2(0) = D_2$,

(ii) $r_2(t)$, $t \in (-\delta\ ,\delta\ )$, are defining functions for $D_2(t)$ that depend smoothly on t.

Suppose, for $t = 0$, there is a minimal lagrangian diffeomorphism $\psi_0 : D_1 \to D_2(0)$. That is, suppose at $t = 0$, there is a solution of (3.7) and consider the question of the existence of solutions to (3.7) for t near 0. We observe that there are no solutions to (3.7) for $t \neq 0$ unless $\operatorname{area}(D_2(t)) = \operatorname{area}(D_1)$. We can, however, allow the area of the domains $D_2(t)$ to vary if we replace the system (3.7) by the somewhat more general system:

$$\begin{aligned} u_x v_y - u_y v_x &= a(t), \\ u_y - v_x &= \gamma(t)(u_x + v_y), \\ r_2(t)(u,v) = 0 \quad &\text{if} \quad r_1(x,y) = 0, \end{aligned} \tag{3.8}$$

where

$$a(t) = \frac{\operatorname{area}(D_2(t))}{\operatorname{area}(D_1)}, \quad a(0) = 1.$$

We recover (3.7) when $\operatorname{area}(D_2(t)) = \operatorname{area}(D_1)$. For the remainder of this section we suppose that we have a solution (u,v) of (3.8) at $t = 0$, and consider the existence of solutions to (3.8) for t near 0. For notational convenience we set,

$$x_1 = x, \quad x_2 = -y.$$

The linearization of the system (3.8) at (u,v) is then:

$$\begin{aligned} &\left[\begin{pmatrix} -v_{x_2} & u_{x_2} \\ v_{x_1} - u_{x_2} & -(u_{x_1} + v_{x_2}) \end{pmatrix} \frac{\partial}{\partial x_1} \right. \\ &\left. - \begin{pmatrix} v_{x_1} & -u_{x_1} \\ u_{x_1} + v_{x_2} & -(u_{x_2} - v_{x_1}) \end{pmatrix} \frac{\partial}{\partial x_2} \right] \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \dot{a} \\ \dot{\gamma}(u_{x_1} - v_{x_2})^2 \end{pmatrix}. \end{aligned} \tag{3.9}$$

The linearized boundary condition is:

$$(\nabla r_2) \cdot (\dot{u}, \dot{v}) = -\dot{r}_2(u,v) \quad \text{on} \quad \partial D_1. \tag{3.10}$$

Proposition 3.2. *Let (ξ_1, ξ_2) be isothermal coordinates for the metric induced by f_ψ. The linear boundary system (**??**), (**??**) is equivalent to the linear boundary system*

$$\begin{aligned} U_{\xi_1} - V_{\xi_2} + A_{11}U + A_{12}V &= \dot{a}\ f_{11} + \dot{\gamma}\ f_{12}, \\ V_{\xi_1} + U_{\xi_2} + A_{21}U + A_{22}V &= \dot{a}\ f_{21} + \dot{\gamma}\ f_{22}, \end{aligned} \tag{3.11}$$

$$RU + SV = -\dot{r}_2 \quad on \quad \partial D_1 \tag{3.12}$$

for some C^∞ functions f_{jk}, $j, k = 1, 2$, on D_1, where R and S are C^∞ functions on ∂D_1 which satisfy:

(i) The vector $(R, S) \in \mathbb{R}^2$ is everywhere non-zero along ∂D_1,

(ii) The smooth map $S^1 \simeq \partial D_1 \to \mathbb{R}^2 \setminus \{0\}$ determined by (R, S) has winding number -1 with respect to the orientations given by (ξ_1, ξ_2) and (U, V) on $\mathbb{R}^2$.

Proof. Left to the reader q.e.d.

Set

$$\begin{aligned} &P(U, V) \\ &= (U_{\xi_1} - V_{\xi_2} + A_{11}U + A_{12}V,\ V_{\xi_1} + U_{\xi_2} + A_{21}U + A_{22}V) \\ &B(U, V) \\ &= RU + SV \quad \text{on} \quad \partial D_1. \end{aligned} \tag{3.13}$$

Theorem 3.3. *The linear boundary system (**??**), (**??**) is elliptic (in the sense of Hörmander [**?**, §20.1]) and hence the operator $(P(U, V), B(U, V))$ is Fredholm on suitable Sobolev spaces.*

Proof. The fact that (**??**) is elliptic is clear. It is then a straightforward computation to show that at a boundary point $p \in \partial D_1$ the boundary condition (**??**) is elliptic if (and only if) the vector $(R, S)(p) \neq 0$. The result follows. q.e.d.

To compute the index of $(P(U, V), B(U, V))$ we first simplify the problem by making a conformal diffeomorphism $D \to D_1$, where D is the unit disc in $\mathbb{R}^2$, and transforming $(P(U, V), B(U, V))$ by this map. Since the transformation is conformal, the form of $(P(U, V), B(U, V))$ remains unchanged. Set,

$$\begin{aligned} W &= U + iV, \qquad \zeta = \xi_1 + i\xi_2, \\ \overline{A_1} &= \tfrac{1}{4}(A_{11} + iA_{21} - A_{12} - iA_{22}), \\ A_2 &= \tfrac{1}{4}(A_{11} + iA_{21} + A_{12} + A_{22}). \end{aligned} \tag{3.14}$$

Then $P(U,V)$ can be written as

$$P(W) = \frac{\partial W}{\partial \bar{\zeta}} + \overline{A}_1 W + A_2 \overline{W}. \tag{3.15}$$

Set, $e^{i\sigma} = \dfrac{(R-iS)}{\sqrt{R^2+S^2}}$ on ∂D. Then we can write $B(U,V)$ as

$$B(W) = \mathrm{Re}(e^{i\sigma} \cdot W). \tag{3.16}$$

Theorem 3.4. *The Riemann-Hilbert boundary system*

$$\begin{aligned} \frac{\partial W}{\partial \bar{\zeta}} + \overline{A_1} W + A_2 \overline{W} &= \dot{a} F_1 + \dot{\gamma} F_2 \quad on \quad D \\ Re(e^{i\sigma} \cdot W) &= \frac{-\dot{r}_2}{\sqrt{R^2+S^2}} \quad on \quad \partial D, \end{aligned} \tag{3.17}$$

has index $= -1$, *where* $F_j = \frac{1}{2}(f_{1j} + i f_{2j}), \quad j = 1,2$. *The kernel of the system is zero and therefore the dimension of the cokernel is one.*

Proof. Let $\Delta \arg$ denote the change in the argument around ∂D. Then it is well known [?] that the index of Riemann-Hilbert boundary systems is given by:

$$\text{index} = 1 - \frac{1}{\pi} \Delta \arg(e^{i\sigma}). \tag{3.18}$$

The winding number of (R,S) considered as a map $S^1 \simeq \partial D_1 \to \mathbb{R}^2 \backslash \{0\}$ is -1. Since $\Delta \arg(e^{i\sigma}) = \Delta \arg(R - iS)$, we have $\Delta \arg(e^{i\sigma}) = 2\pi$. The results on the kernel and cokernel are also standard [?]. q.e.d.

We conclude that the boundary system (**??**), (**??**) also has index $= -1$, zero kernel and cokernel of dimension equal to one.

Since the cokernel has dimension one, there is one condition on the right-hand side of (**??**), that is both necessary and sufficient for the existence of a solution of (**??**). To express this condition consider the adjoint operator to the boundary system (**??**), (**??**). Following [?] the adjoint operator is:

$$\begin{aligned} P^*(Z) =& \frac{\partial Z}{\partial \zeta} - \overline{A_1} Z + \overline{A_2 Z}, \\ B^*(Z) =& \mathrm{Re}(-i e^{-i\sigma} \frac{d\zeta}{ds} Z) \big|_{\partial D}. \end{aligned} \tag{3.19}$$

Proposition 3.5. *The necessary and sufficient condition for the existence of a smooth solution of (**??**) is that*

$$\oint_{\partial D} \frac{\dot{r}_2}{\sqrt{R^2+S^2}} Im(ie^{-i\sigma}\frac{d\zeta}{ds}Z)ds$$

$$= i\iint_D \{-(\dot{a}F_1 + \dot{\gamma}F_2)Z + (\dot{a}\overline{F}_1 + \dot{\gamma}\overline{F}_2)\overline{Z}\}d\xi_1 d\xi_2 \tag{3.20}$$

for all solutions Z *of* $P^*(Z) = 0,\ B^*(Z) = 0$.

Proof. See [**?**] Chapter 1. q.e.d.

The adjoint system (**??**) has one-dimensional kernel. Thus for (**??**) to have a (unique) solution, $(\dot{a}F_1 + \dot{\gamma}F_2, \frac{-\dot{r}_2}{\sqrt{R^2+S^2}})$ must satisfy the one condition imposed by (**??**). It is clear that there is a unique value of the constant $\dot{\gamma}$ (depending on $F_1, F_2, \dot{a}, \dot{r}_2$ and the boundary system (**??**)) such that (**??**) is satisfied.

Theorem 3.6. *There is one (and only one) value of the constant* $\dot{\gamma}$ *(depending on* (u, v) *and their derivatives,* $\nabla r_2, \dot{r}_2$ *and* $\dot{a}$*) such that the linear boundary system (3.9), (3.10) has a unique smooth solution on* D_1.

Returning to the question of finding solutions of (3.8) for t near 0 we have:

Theorem 3.7. *There is an* $\varepsilon > 0$ *such that if* $|t| < \varepsilon$*, then there is a smooth solution of (3.8) on* D_1.

Proof. The result follows from Theorem **??**, the inverse function theorem for Banach spaces and standard elliptic regularity results. We leave the details to the reader. q.e.d.

Applying the theorem to the case where $\text{area}(D_2(t)) = \text{area}(D_1)$ we have:

Corollary 3.8. *There is an* $\varepsilon > 0$ *such that if* $|t| < \varepsilon$*, then there is a minimal lagrangian diffeomorphism* $\psi_t : D_1 \to D_2(t)$.

Remark 3.1. The constant $\dot{\gamma}$ which occurs in Theorem **??** and throughout this section has a geometric interpretation. To understand this we first describe the space of orthogonal complex structures on $\mathbb{R}^4$. Let $\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2}\}$ be an orthonormal frame on $\mathbb{R}^4 \simeq \mathbb{R}^2 \times \mathbb{R}^2$. In terms of this frame we define three complex structures $J_k,\ k = 1, 2, 3$, as follows:

$$
\begin{aligned}
&J_1 : \tfrac{\partial}{\partial x_1} \mapsto \tfrac{\partial}{\partial y_1}, \quad \tfrac{\partial}{\partial y_1} \mapsto -\tfrac{\partial}{\partial x_1}; \quad \tfrac{\partial}{\partial x_2} \mapsto -\tfrac{\partial}{\partial y_2}, \quad \tfrac{\partial}{\partial y_2} \mapsto \tfrac{\partial}{\partial x_2},\\
(3.21) \qquad &J_2 : \tfrac{\partial}{\partial x_1} \mapsto \tfrac{\partial}{\partial y_2}, \quad \tfrac{\partial}{\partial y_2} \mapsto -\tfrac{\partial}{\partial x_1}; \quad \tfrac{\partial}{\partial y_1} \mapsto -\tfrac{\partial}{\partial x_2}, \quad \tfrac{\partial}{\partial x_2} \mapsto \tfrac{\partial}{\partial y_1},\\
&J_3 : \tfrac{\partial}{\partial x_1} \mapsto -\tfrac{\partial}{\partial x_2}, \quad \tfrac{\partial}{\partial x_2} \mapsto \tfrac{\partial}{\partial x_1}; \quad \tfrac{\partial}{\partial y_1} \mapsto -\tfrac{\partial}{\partial y_2}, \quad \tfrac{\partial}{\partial y_2} \mapsto \tfrac{\partial}{\partial y_1}.
\end{aligned}
$$

Note that J_1 is the "standard" complex structure on $\mathbb{C} \times \bar{\mathbb{C}} \simeq \mathbb{R}^4$. The space of all orthogonal complex structures on $\mathbb{R}^4$ forms a two-sphere given by:

$$\mathcal{J} = \{a_1 J_1 + a_2 J_2 + a_3 J_3 : a_k \in \mathbb{R},\ a_1^2 + a_2^2 + a_3^2 = 1\}.$$

If J_1 is the "north pole" of this two-sphere, then $-J_1$ is the "south pole" and the equator is given by:

$$\mathcal{J}_0 = \{a_2 J_2 + a_3 J_3 : a_2^2 + a_3^2 = 1\} \subseteq \mathcal{J}.$$

The symplectic form ω, determined by J_1 and the euclidean metric, is $\omega = dx_1 \wedge dy_1 - dx_2 \wedge dy_2$. Let $\psi : D_1 \to D_2$ be a minimal lagrangian diffeomorphism. Then the surface $S = graph(\psi)$ is both minimal and ω-lagrangian. In particular, it is ω-lagrangian with constant lagrangian angle β. We have,

Proposition 3.9. *$S = graph(\psi)$ is a J-complex curve, for $J = a_2 J_2 + a_3 J_3 \in \mathcal{J}_0$, where $a_2 = sin(\pi\beta)$, $a_3 = cos(\pi\beta)$. In particular, $\gamma = -\frac{a_3}{a_2}$.*

Proof. The tangent space, T_*S, of S is spanned by:

$$
\begin{aligned}
X &= \tfrac{\partial}{\partial x_1} + u_x \tfrac{\partial}{\partial x_2} + v_x \tfrac{\partial}{\partial y_2},\\
Y &= \tfrac{\partial}{\partial y_1} + u_y \tfrac{\partial}{\partial x_2} + v_y \tfrac{\partial}{\partial y_2}.
\end{aligned}
$$

Using (3.21) and the equation $u_x v_y - u_y v_x = 1$ we have that $\text{span}(X, Y)$ is $J = a_2 J_2 + a_3 J_3$ invariant if and only if: $a_3(u_x + v_y) = a_2(v_x - u_y)$. Hence, from (3.6) it follows that, $\gamma = -\frac{a_3}{a_2}$. Therefore, $a_2 = \sin(\pi\beta)$, $a_3 = \cos(\pi\beta)$. q.e.d.

Consider a family of minimal lagrangian diffeomorphisms $\psi_t : D_1 \to D_2(t)$ with D_1 and $D_2(t)$ as described above. Then the surfaces $\text{graph}(\psi_t)$ are J_t-complex for $J_t \in \mathcal{J}_0$. The family $\{J_t\}$ is determined by the functions $\beta(t)$ and hence by the functions $\gamma(t)$. Since $\dot{\gamma}$ is the derivative of γ with respect to t, we see that the local existence problem for minimal lagrangian diffeomorphisms is solvable because the set of complex structures $\mathcal{J}_0$ is one-dimensional. This parameter allows the cokernel condition (**??**) to be satisfied.

4. The Monge-Ampère equation and an a priori estimate

Recall the formulation of the equations of a minimal lagrangian diffeomorphism ψ given in (3.7):

$$\begin{array}{llcl} (a) & u_x v_y - u_y v_x & = & 1, \\ (b) & \sin(\pi\beta)(u_y - v_x) & = & -\cos(\pi\beta)(u_x + v_y), \\ & r_2(u,v) = 0 & \text{if} & r_1(x,y) = 0, \end{array} \tag{4.1}$$

where the lagrangian angle β is constant. Given the minimal lagrangian diffeomorphism ψ we can compute the lagrangian angle along its graph using any parallel unit $(2,0)$ form σ. Choosing $\sigma = e^{i\pi\theta} dz_1 \wedge d\bar{z}_2$ gives, for different choices of θ, different values of β in (**??**). Thus we can take for β in (**??**) any constant we choose. In particular, choose:

$$\beta = \tfrac{1}{2}. \tag{4.2}$$

Then (**??**) becomes:

$$\begin{array}{llcl} (a) & u_x v_y - u_y v_x & = & 1, \\ (b) & u_y & = & v_x, \\ & r_2(u,v) = 0 & \text{if} & r_1(x,y) = 0. \end{array} \tag{4.3}$$

Since D_1 is simply connected, from (**??**b), it follows that there is a smooth real-valued function w on D_1 such that:

$$w_x = u, \quad w_y = v. \tag{4.4}$$

It is then easy to verify that (**??**a) becomes:

$$w_{xx} w_{yy} - (w_{xy})^2 = 1. \tag{4.5}$$

A solution of (4.1) thus yields a convex function w on D_1 satisfying the Monge-Ampère equation (4.5) such that ∇w defines a diffeomorphism $D_1 \to D_2$. That is, a solution of (4.1) gives a solution of the *the second boundary-value problem for the Monge-Ampère equation* for the domains (D_1, D_2).

It is clear from (4.1) that the gradient of a solution of the second boundary-value problem for the domains (D_1, D_2) is a minimal lagrangian diffeomorphism $D_1 \to D_2$. Thus from Theorem **??** we have:

Theorem 4.1. *Let (D_1, D_2) be a pair of domains in $\mathbb{R}^2$ with smooth boundaries and equal areas. If*

$$\mathit{variation}(D_1, D_2) \geq 1, \tag{4.6}$$

then there is no regular solution of the second boundary-value problem for the Monge-Ampère equation.

The regularity of the solution of the second boundary-value problem has not been extensively investigated without convexity assumptions on both domains. However Caffarelli [?] has given an example of a nonconvex domain in $\mathbb{R}^2$ with unit area such that there is no regular solution of (4.5) whose gradient defines a diffeomorphism from the unit disc into this domain. He remarks that the conditions needed on the domains to insure regularity are of a geometrical rather than a topological or differential nature. Using Proposition 2.2 it is not difficult to verify that Caffarelli's example satisfies (4.6). In light of this the following questions are appropriate:

Question. Let (D_1, D_2) be a pair of domains in $\mathbb{R}^2$ with smooth boundaries and equal areas, satisfying

$$variation(D_1, D_2) < 1.$$

Does there exist a minimal lagrangian diffeomorphism $\psi : D_1 \to D_2$ smooth up to the boundary? Equivalently, does there exist a smooth solution of the second boundary-value problem for the Monge-Ampère equation?

The work of Delanoë [?] and Caffarelli [?] [?] gives an affirmative answer to both questions in case both domains are strictly convex. The remainder of this paper is devoted to giving a more complete answer though, in general, the questions remain open.

The system (4.1) can be interpreted in yet another way. We have already shown, in the notation of Remark 3.3, that a minimal lagrangian diffeomorphism ψ has a graph which is J-complex for some $J \in \mathcal{J}_0$. Thus the map,

$$\begin{array}{rll} f_\psi : & (D_1, \partial D_1) & \to (\mathbb{R}^4, T^2), \\ & (x, y) & \mapsto (x, y, \psi(x, y)), \end{array} \tag{4.7}$$

is minimal with image a J-holomorphic curve. Let D denote the unit disc in $\mathbb{R}^2$ centered at the origin. Consider D_1 with the conformal structure determined by the metric induced by f_ψ. Let $\phi : D \to D_1$ be a conformal diffeomorphism.

Lemma 4.2. *The map*

$$F_\psi = f_\psi \circ \phi : (D, \partial D) \to (\mathbb{R}^4, T^2)$$

is $J-$holomorphic.

Proof. Left to the reader. q.e.d.

The maps f_ψ and F_ψ have some interesting and useful properties.

Proposition 4.3. *If f_ψ is the minimal map in (4.7), and one of the domains D_i, $i = 1, 2$, is strictly convex, then $area(f_\psi) < A$ where A depends on the geometry of ∂D_1 and ∂D_2, but is independent of f_ψ.*

Proof. This follows from the isoperimetric inequality for minimal discs in $\mathbb{R}^n$ and Theorem **??** q.e.d.

Let r_i be a defining function for D_i, $\quad i = 1, 2$. That is, suppose

$$D_i = \{(s,t) \in \mathbb{R}^2 : r_i(s,t) \leq 0\}, \tag{4.8}$$

and $\nabla r_i \neq 0$ along ∂D_i. Consider the hessian of r_i, $\text{Hess}(r_i)$, on D_i. Let σ_i denote the minimum value of the eigenvalues of $\text{Hess}(r_i)$ on D_i.

Definition 4.1. We say the pair (r_1, r_2) is *pseudoconvex* if:

$$\sigma_1 + \sigma_2 > 0. \tag{4.9}$$

Definition 4.2. We say the pair (D_1, D_2) is *pseudoconvex* if the domains admit a pair of pseudoconvex defining functions.

Proposition 4.4. *Let κ_i denote the curvature of ∂D_i in $\mathbb{R}^2$. The pair (D_1, D_2) is pseudoconvex if and only if*

$$\min_{\partial D_1} \kappa_1 + \min_{\partial D_2} \kappa_2 > 0.$$

Proof. Left to the reader. q.e.d.

We justify the use of the term *pseudoconvex* as follows: Let (D_1, D_2) be a pair of domains with defining functions (r_1, r_2). Set

$$\begin{aligned} &r : \mathbb{R}^4 \to \mathbb{R}, \\ &r(x_1, y_1, x_2, y_2) = r_1(x_1, y_1) + r_2(x_2, y_2). \end{aligned} \tag{4.10}$$

Proposition 4.5. *If (r_1, r_2) is pseudoconvex, then for every $J \in \mathcal{J}_0$ the function r is strictly J-pseudoconvex in an open neighborhood of the domain $D_1 \times D_2 \subset \mathbb{R}^4$.*

Proof. Recall the complex structures J_k, $k = 1, 2, 3$, and the description of the space of all orthogonal complex structures $\mathcal{J}$ on $\mathbb{R}^4$ given in Remark (3.3). A unitary frame of the $J_k + i$-eigenspace is given by,

$$
\begin{aligned}
J_1 &: \{\tfrac{\partial}{\partial z_1} = \tfrac{1}{\sqrt{2}}(\tfrac{\partial}{\partial x_1} - i\tfrac{\partial}{\partial y_1}),\ \tfrac{\partial}{\partial z_2} = \tfrac{1}{\sqrt{2}}(\tfrac{\partial}{\partial x_2} + i\tfrac{\partial}{\partial y_2})\},\\
J_2 &: \{\tfrac{1}{\sqrt{2}}(\tfrac{\partial}{\partial x_1} - i\tfrac{\partial}{\partial y_2}),\ \tfrac{1}{\sqrt{2}}(\tfrac{\partial}{\partial y_1} + i\tfrac{\partial}{\partial x_2})\},\\
J_3 &: \{\tfrac{1}{\sqrt{2}}(\tfrac{\partial}{\partial x_1} + i\tfrac{\partial}{\partial x_2}),\ \tfrac{1}{\sqrt{2}}(\tfrac{\partial}{\partial y_1} + i\tfrac{\partial}{\partial y_2})\}.
\end{aligned}
$$

Let $J = a_2J_2 + a_3J_3 \in \mathcal{J}_0$ be a complex structure where ${a_2}^2 + {a_3}^2 = 1$. A straightforward computation shows that a unitary frame of the J $+i$-eigenspace is given by,

$$
\begin{aligned}
\tfrac{\partial}{\partial \xi_1} &= \tfrac{1}{\sqrt{2}}(\tfrac{\partial}{\partial x_1} + ia_3\tfrac{\partial}{\partial x_2} - ia_2\tfrac{\partial}{\partial y_2}),\\
\tfrac{\partial}{\partial \xi_2} &= \tfrac{1}{\sqrt{2}}(\tfrac{\partial}{\partial y_1} + ia_2\tfrac{\partial}{\partial x_2} + ia_3\tfrac{\partial}{\partial y_2}).
\end{aligned}
$$

r is a strictly J-pseudoconvex function if

$$
\sum_{\ell,k} \frac{\partial^2 r}{\partial \xi_\ell \partial \bar{\xi}_k}\, \nu_\ell \bar{\nu}_k > 0 \ \text{ for } \ (\nu_1, \nu_2) \neq (0,0). \tag{4.11}
$$

We have,

$$
\begin{aligned}
\sum_{\ell,k} \frac{\partial^2 r}{\partial \xi_\ell \partial \bar{\xi}_k}\nu_\ell \bar{\nu}_k &= \sum_{\ell,k} \frac{\partial^2 r_1}{\partial \xi_\ell \partial \bar{\xi}_k}\nu_\ell \bar{\nu}_k + \sum_{\ell,k} \frac{\partial^2 r_2}{\partial \xi_\ell \partial \bar{\xi}_k}\nu_\ell \bar{\nu}_k.\\
&\geq \frac{1}{2}(\sigma_1 + \sigma_2)(|\nu_1|^2 + |\nu_2|^2),
\end{aligned} \tag{4.12}
$$

where the σ_i denote the minimum value of the eigenvalues of $\mathrm{Hess}(r_i)$ on D_i. The result follows. q.e.d.

Suppose that the pair (D_1, D_2) is pseudoconvex with pseudoconvex defining functions (r_1, r_2). Suppose further that $\psi : D_1 \to D_2$ is a minimal lagrangian diffeomorphism with minimal map $f_\psi : (D_1, \partial D_1) \to (\mathbb{R}^4, T^2)$. Using Proposition 4.5 we have that $r = r_1 + r_2$ is strictly $J-$pseudoconvex near $D_1 \times D_2$. By perturbing r, if necessary, outside a neighborhood of $D_1 \times D_2$ we can suppose without loss of generality that $r^{-1}(0)$ is a smooth compact strictly $J-$pseudoconvex hypersurface containing $T^2 = \partial D_1 \times \partial D_2$ and bounding a $J-$pseudoconvex domain $W \subset \mathbb{R}^4$. Moreover we can suppose that $\nabla r_{|\partial W}$ is everywhere nonzero and outward pointing.

As above, let D be the unit disc in $\mathbb{R}^2$ centered at the origin. The next proposition and its corollary use the pseudoconvexity of r.

Proposition 4.6. *If $h : (D, \partial D) \to (W, T^2)$ is a J-holomorphic map and $h \in C^1(D)$, then the image of the interior of D lies in the interior of W and, for every $x \in \partial D$,*

$$(\nabla r \cdot \frac{\partial h}{\partial \nu})(x) > 0,$$

where $\frac{\partial h}{\partial \nu}(x)$ is the normal derivative of h at x.

Proof. Follows from the J-pseudoconvexity of r and the Hopf boundary point lemma. q.e.d.

Corollary 4.7. *If $h : (D, \partial D) \to (W, T^2)$ is a J-holomorphic map in $C^1(D)$, then the boundary curve of h lies in the set of totally real points of T^2.*

Proof. Let $x \in \partial D$. Suppose that $h(x)$ is a complex tangent point, i.e., $T_{h(x)}(T^2)$ is a complex line. Since $h_*(T_x D)$ is a complex line and $h_*(T_x D)$ and $T_{h(x)}(T^2)$ intersect in the real line $h_*(T_x(\partial D))$, they must coincide. Hence $\frac{\partial h}{\partial \nu}(x)$ is tangent to $T^2 \subset \partial W$, contradicting the previous proposition. q.e.d.

The corollary applied to the J-holomorphic map F_ψ shows that the boundary trace of f_ψ (or ψ) misses the J-complex tangent points on T^2. The next theorem is a refined version of Proposition 4.6.

Theorem 4.8. *Suppose that (D_1, D_2) is pseudoconvex with pseudoconvex defining functions (r_1, r_2). Let $\psi : D_1 \to D_2$ be a minimal lagrangian diffeomorphism. Set $f_\psi = id \times \psi : D_1 \to \mathbb{R}^4$ and $r = r_1 + r_2$. Then there is a constant $c > 0$ depending on (D_1, D_2), but independent of ψ, such that at any point on ∂D_1 :*

$$\nabla r \cdot \frac{\partial f_\psi}{\partial \nu_1} \geq c > 0,$$

where $\frac{\partial}{\partial \nu_1}$ denotes the normal derivative on ∂D_1.

Proof. Suppose that β is chosen (as in (4.2)) such that the equations for the minimal lagrangian diffeomorphism ψ become the Monge-Ampère equation for the convex function w on D_1. Set

$$x_1 = x, \quad x_2 = y.$$

Then w satisfies:

$$w_{x_1} = u, \quad w_{x_2} = v, \tag{4.13}$$

$$w_{x_1x_1}w_{x_2x_2} - (w_{x_1x_2})^2 = 1. \tag{4.14}$$

For brevity of notation write $w_{x_i} = w_i, \qquad w_{x_ix_j} = w_{ij}$, etc. Set,

$$(w^{ij}) = (w_{ij})^{-1}.$$

Differentiating (4.14) with respect to x_k we get,

$$\sum_{i,j} w^{ij}w_{ijk} = 0. \tag{4.15}$$

Consider the function R on D_1 given by,

$$R = r_1(x_1, x_2) + r_2(u, v). \tag{4.16}$$

For any $a \in (0, \infty)$ consider $R - ar_1$ on D_1 and compute $\sum_{i,j} w^{ij}(R - ar_1)_{ij}$:

$$\begin{aligned}
&\sum_{i,j} w^{ij}(R - ar_1)_{ij} \\
&\qquad = \sum_{i,j} w^{ij}(r_2(w_1, w_2) + (1-a)r_1(x_1, x_2))_{ij} \\
&\qquad = \sum_{i,j}(w_{ij}(r_2)_{ij} + (1-a)w^{ij}(r_1)_{ij}) \\
&\qquad = \sum_{i,j} w^{ij}(\det((r_2)_{ij})(r_2)^{ij} + (1-a)(r_1)_{ij}).
\end{aligned} \tag{4.17}$$

The second equality follows from (4.15). The eigenvalues of the matrix

$$\det((r_2)_{ij})(r_2)^{ij}$$

are the same as those of the matrix $(r_2)_{ij} = \mathrm{Hess}(r_2)$. Because the pair (r_1, r_2) is pseudoconvex, there is some $a > 0$ such that both eigenvalues of the matrix,

$$\det((r_2)_{ij})(r_2)^{ij} + (1-a)(r_1)_{ij}$$

are positive. Since w^{ij} is positive definite, it follows from (4.17) that

$$\sum_{i,j} w^{ij}(R - ar_1)_{ij} > 0. \tag{4.18}$$

Clearly, $R - ar_1 = 0$ on ∂D_1. Thus by the Hopf maximum principle at any point of ∂D_1,

$$\frac{\partial R}{\partial \nu_1} > a\frac{\partial r_1}{\partial \nu_1} \geq c > 0. \tag{4.19}$$

From (**??**) we have $R = r \circ f_\psi$. The result follows. q.e.d.

Suppose that $\psi : D_1 \to D_2$ is a minimal lagrangian diffeomorphism. Then $\psi^{-1} : D_2 \to D_1$ is also. Let $p \in \partial D_1,\ \ q = \psi(p) \in \partial D_2$.

Lemma 4.9. *If* $|\frac{\partial \psi}{\partial \nu_1}(p)| \geq 1$, *then* $|\frac{\partial \psi^{-1}}{\partial \nu_2}(q)| \leq 1$ *and conversely, where* $\frac{\partial}{\partial \nu_i}$ *is the normal derivative along* ∂D_i.

Proof. Choose euclidean coordinates (x, y) such that at $p \in \partial D_1$,

$$\begin{aligned} \tfrac{\partial}{\partial x} &= \text{unit normal to } \partial D_1, \\ \tfrac{\partial}{\partial y} &= \text{unit tangent to } \partial D_1. \end{aligned}$$

At $q \in \partial D_2$ choose euclidean coordinates (u, v) such that,

$$\begin{aligned} \tfrac{\partial}{\partial u} &= \text{unit normal to } \partial D_2, \\ \tfrac{\partial}{\partial v} &= \text{unit tangent to } \partial D_2. \end{aligned}$$

With respect to these coordinates $\psi(x, y) = (u, v)$ satisfies (**??**). In particular,

$$u_x v_y - u_y v_x = 1. \tag{4.20}$$

(Equivalently, $\psi^{-1}(u, v) = (x, y)$ satisfies $x_u y_v - x_v y_u = 1$.) The boundary condition implies,

$$u_y(p) = 0. \tag{4.21}$$

Using (**??**b) it follows that,

$$v_x(p) = 0. \tag{4.22}$$

(Equivalently, $y_u(q) = 0$.) Thus combining (**??**) and (**??**) gives,

$$u_x(p) v_y(p) = 1. \tag{4.23}$$

(Equivalently, $x_u(q) y_v(q) = 1$.) Since ψ and ψ^{-1} are inverses their Jacobian matrices satisfy:

$$\begin{pmatrix} v_y & -v_x \\ -u_y & u_x \end{pmatrix}(p) = \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix}^{-1}(p) = \begin{pmatrix} x_u & y_u \\ x_v & y_v \end{pmatrix}(q). \tag{4.24}$$

Using (**??**) and (**??**) we have,

$$\frac{\partial\psi}{\partial\nu_1}(p) = \frac{\partial}{\partial x}(u,v)(p) = (u_x, v_x)(p) = (u_x(p), 0),$$

$$\frac{\partial\psi^{-1}}{\partial\nu_2}(q) = \frac{\partial}{\partial u}(x,y)(q) = (x_u(q), 0) = (v_y(p), 0).$$

Hence, by (**??**),

$$|\frac{\partial\psi}{\partial\nu_1}(p)||\frac{\partial\psi^{-1}}{\partial\nu_2}(q)| = 1. \tag{4.25}$$

The result follows. q.e.d.

Consider the maps $f_\psi = \mathrm{id} \times \psi : D_1 \to \mathbb{R}^4$ and

$$f_{\psi^{-1}} = \psi^{-1} \times \mathrm{id} : D_2 \to \mathbb{R}^4.$$

Clearly they have the same graphs. Moreover,

$$\begin{aligned} |\frac{\partial f_\psi}{\partial\nu_1}|^2 &= 1 + |\frac{\partial\psi}{\partial\nu_1}|^2, \\ |\frac{\partial f_{\psi^{-1}}}{\partial\nu_2}|^2 &= |\frac{\partial\psi^{-1}}{\partial\nu_2}|^2 + 1. \end{aligned}$$

Hence if $q = \psi(p)$ then, by the lemma,

$$\text{if } |\frac{\partial f_\psi}{\partial\nu_1}(p)|^2 \geq 2 \text{ then } |\frac{\partial f_{\psi^{-1}}}{\partial\nu_2}(q)|^2 \leq 2, \tag{4.26}$$

and conversely. Therefore we have:

Proposition 4.10. *Let $(p,q) \in \partial D_1 \times \partial D_2$ with $q = \psi(p)$. Let $\theta(p,q)$ be the angle between the planes $T_{(p,q)}(T^2)$ and*

$$(f_\psi)_* T_p(D_1) = (f_{\psi^{-1}})_* T_q(D_2).$$

Then

$$\theta(p,q) \geq \delta > 0,$$

where δ depends on $|\nabla r|$ and the geometry of ∂D_i but is independent of ψ and the point (p,q).

Proof. By Theorem 4.8 we have at $(p,q) \in T^2$,

$$|\nabla r||\frac{\partial f_\psi}{\partial \nu_1}| \quad \cos \quad \rho = \nabla r \cdot \frac{\partial f_\psi}{\partial \nu_1} \geq c,$$

where ρ is the angle between ∇r and $\frac{\partial f_\psi}{\partial \nu_1}$. We can suppose that

$$|\frac{\partial f_\psi}{\partial \nu_1}(p)| \leq \sqrt{2},$$

since otherwise we consider $\frac{\partial f_{\psi^{-1}}}{\partial \nu_2}(q)$. Thus,

$$\cos \quad \rho \geq \frac{c}{\sqrt{2}|\nabla r|} > 0.$$

This implies that the angle ρ satisfies,

$$0 \leq \rho < \frac{\pi}{2} - \delta,$$

where $\delta > 0$ depends on c and $|\nabla r|$. Since ∇r is normal to $T_{(p,q)}(T^2)$, the angle between $T_{(p,q)}(T^2)$ and ${f_\psi}_* T_p(D_1)$ is $\geq \delta$. q.e.d.

Theorem 4.11. *Let (D_1, D_2) be a pseudoconvex pair and let $\psi : D_1 \to D_2$ be a minimal lagrangian diffeomorphism smooth up to the boundary. Let $J \in \mathcal{J}_0$ denote the complex structure such that graph (ψ) is J-holomorphic. Then the distance between the boundary trace of ψ on $T^2 = \partial D_1 \times \partial D_2$ and the J-complex tangent points on T^2 is bounded away from zero by a constant depending on $|\nabla r|$ and the geometry of ∂D_1 and ∂D_2 but independent of ψ and J.*

Proof. Fix $J \in \mathcal{J}_0$ and consider the set of minimal lagrangian diffeomorphisms:

$$\mathcal{S}_J = \{\psi : D_1 \to D_2 : \text{graph}(\psi) \text{ is } J-\text{holomorphic}\}.$$

Suppose the boundary traces of diffeomorphisms $\psi \in \mathcal{S}_J$ are not bounded away from the complex tangent points of J. Then there is a sequence $\{\psi_\nu\}$ of maps in $\mathcal{S}_J$ with boundary trace approaching a J-complex tangent point, $x \in T^2$. In particular on each boundary trace there is a point $x_\nu \in T^2$ with $x_\nu \to x$. Denote the tangent space to graph(ψ) at x_ν by P_{x_ν}. For each ν, the 2-plane P_{x_ν} intersects the 2-plane $T_{x_\nu}(T^2)$ in a real line L_{x_ν}. Since P_{x_ν} is a J-complex line,

$$P_{x_\nu} = L_{x_\nu} \wedge J L_{x_\nu}. \tag{4.27}$$

Because x is a J-complex tangent point, as $x_\nu \to x$ the distance between $T_{x_\nu}(T^2)$ and $L_{x_\nu} \wedge JL_{x_\nu}$ goes to zero. Hence, by (**??**), $T_{x_\nu}(T^2)$ becomes arbitrarily close to P_{x_ν}. This contradicts Proposition **??**. Thus the boundary traces of diffeomorphisms $\psi \in \mathcal{S}_J$ are bounded away from the J-complex tangent points on T^2 by a bound depending on $J, |\nabla r|, \partial D_1$ and ∂D_2.

Now repeat this argument for each $J \in \mathcal{J}_0$. Since $\mathcal{J}_0$ is compact the result follows. q.e.d.

5. Existence: the continuity method

In this section we prove:

Theorem 5.1. *Let (D_1, D_2) be a pseudoconvex pair of domains with smooth boundaries, satisfying area(D_1) = area(D_2). Then there is an area-preserving diffeomorphism $\psi : D_1 \to D_2$, smooth up to the boundary with graph a minimal surface in $\mathbb{R}^4 = \mathbb{C}^2$.*

Corollary 5.2. *Let (D_1, D_2) be a pseudoconvex pair of domains with smooth boundaries, satisfying area(D_1) = area(D_2). Then there is a smooth solution of the second boundary value problem for the Monge-Ampère equation. That is, there is a smooth function w satisfying,*

$$w_{xx}w_{yy} - (w_{xy})^2 = 1$$

such that the gradient of w, ∇w, defines a diffeomorphism

$$(D_1, \partial D_1) \to (D_2, \partial D_2).$$

The proof of the theorem uses the continuity method as follows: Since (D_1, D_2) is a pseudoconvex pair, at least one of the domains is strictly convex. Without loss of generality we can suppose that D_1 is strictly convex with strictly convex defining function r_1. Let $D_2(t)$, $0 \le t \le 1$ be a smooth (in t) family of domains in $\mathbb{R}^2$ with smooth boundary and with defining functions $r_2(t)$, $0 \le t \le 1$, satisfying the following:

(i) For each t,the pair $(D_1, D_2(t))$ is pseudoconvex and area $(D_2(t))$ = area (D_1).

(ii) For each t, the functions $r_2(t)$ vary smoothly in t and the pair $(r_1, r_2(t))$ is pseudoconvex.

(iii) $D_2(0) = D_1$ and $D_2(1) = D_2$.

For each t we seek an area-preserving diffeomorphism $\psi_t : D_1 \to D_2(t)$ smooth up to the boundary whose graph is a minimal surface. The ψ_t are minimal lagrangian diffeomorphisms. Clearly, when $t = 0$ we can take $\psi_0 = id$. The continuity method requires we show that the set of t, such that ψ_t exists, is both open and closed. This shown, it follows that $\psi_1 : D_1 \to D_2$ exists and this proves the theorem.

Openness. Openness follows immediately from the "inverse function theorem" Corollary **??**.

Closedness. We show that the set of t, for which there exists a minimal lagrangian diffeomorphism $\psi_t : D_1 \to D_2(t)$ (smooth up to the boundary), is closed. To do this we suppose that for each $t < t_0$ there is a minimal lagrangian diffeomorphism $\psi_t : D_1 \to D_2(t)$ depending continuously on t. We must show that there is a minimal lagrangian diffeomorphism $\psi_{t_0} : D_1 \to D_2(t_0)$.

For each t, the ψ_t determine smooth maps $f_t = id \times \psi_t$:

$$f_t : D_1 \to \mathbb{R}^2 \times \mathbb{R}^2 \simeq \mathbb{R}^4. \tag{5.1}$$

The f_t are minimal lagrangian maps. Recall that Proposition 4.3 shows that there is a constant $A(t)$ depending only on the geometry of the domains D_1 and $D_2(t)$ such that for each area$(f_t) \leq A(t)$. Setting $A = \sup_{0 \leq t \leq 1} A(t)$, we have for all t

$$\text{area}(f_t) \leq A. \tag{5.2}$$

Let,

$$r_t(x, y, u, v) = r_1(x, y) + r_2(t)(u, v), \tag{5.3}$$

$$T^2(t) = \partial D_1 \times \partial D_2(t). \tag{5.4}$$

Then, for each t, r_t is strictly $J-$pseudoconvex in a neighborhood of $T^2(t)$ for all $J \in \mathcal{J}_0$. As in §4 we can, by perturbing, assume that, for each t, r_t is strictly $J-$pseudoconvex and that $r_t^{-1}(0)$ is a strictly $J-$pseudoconvex hypersurface containing $T^2(t)$.

The maps f_t, defined in (5.1), are minimal lagrangian maps, and so there is an orthogonal complex structure, $J_t \in \mathcal{J}_0$, such that $image(f_t)$ is a J_t-holomorphic curve. The boundary trace of f_t coincides with the boundary trace of ψ_t. Hence by Theorem 4.11, for each t, the distance between the boundary trace of f_t and the J_t complex tangent points of $T^2(t)$ is bounded away from zero by a constant depending only on $r_t, \partial D_1$ and $\partial D_2(t)$. Using the compactness of $[0,1]$ we can assume this constant to be independent of t. We now conformally reparameterize each map f_t to construct J_t-holomorphic maps from D, the unit disc centered at the origin into $\mathbb{R}^4$,

$$F_t : (D, \partial D) \to (\mathbb{R}^4, T^2(t)).$$

By choosing an appropriate conformal reparameterization we can suppose that for each t :

$$(\star) \qquad r_t(F_t(0)) \leq -1.$$

Since the image of F_t is the same as the image of f_t, the distance between the boundary trace of F_t and the J_t-complex tangent points of $T^2(t)$ is bounded away from zero by a constant independent of F_t and t. Since the reparameterization is conformal,

$$\text{area}(F_t) = \text{area}(f_t).$$

Hence,

$$\text{area}(F_t) \leq A. \tag{5.5}$$

For each t, the complex structure J_t is an element in $\mathcal{J}_0$. Thus we can choose a subsequence of the $\{t\}$ that we denote $\{t_\gamma\}$ such that the J_{t_γ} converge smoothly to an orthogonal complex structure $J_{t_0} \in \mathcal{J}_0$. Consider the sequence, $\{F_{t_\gamma}\}$, of J_{t_γ}-holomorphic maps.

Theorem 5.3. *For any $k \geq 1$, there is a subsequence of $\{F_{t_\gamma}\}$ (which we still denote $\{F_{t_\gamma}\}$) which converges in $C^k(D)$ to a $J_{t_0}-$holomorphic map*

$$F_{t_0} : (D, \partial D) \to (\mathbb{R}^4, T^2(t_0)).$$

The boundary of F_{t_0} is a smooth $(1,1)$ curve on the torus

$$T^2(t_0) = \partial D_1 \times \partial D_2(t_0).$$

Proof. Since the boundary trace of the maps F_{t_γ} lie in the totally real points of the surface $T^2(t_\gamma)$, the maps satisfy elliptic boundary conditions. Further, since the boundary trace are uniformly bounded away from the complex tangent points, the boundary conditions are uniformly elliptic. Hence we have the standard uniform boundary estimates for $J-$holomorphic maps as in Floer [**?**]. In the interior we have the standard interior elliptic estimates for $J-$holomorphic maps as in [**?**] or [**?**]. The condition ($\star$) insures that the reparameterization group is compact. Combining these estimates with the uniform area bound (5.5) it follows that a subsequence of the maps F_{t_γ} converges in C^k to a J_{t_0} holomorphic map F_{t_0} up to "bubbling" (see for example [**?**]).

We next show that there is no bubbling. Interior bubbling gives nontrivial J_{t_0} holomorphic 2-spheres in $\mathbb{R}^4$. This is clearly impossible. Hence in the interior the convergence is C^k. Recall that the boundary trace of the holomorphic maps F_{t_γ} are uniformly bounded away from the complex tangent points of the surface $T^2(t_\gamma)$. These surfaces lie in the strictly $J_{t_\gamma}-$pseudoconvex hypersurfaces $r_{t_\gamma}^{-1}(0)$. Bedford-Gaveau [**?**] derive uniform Lipshitz estimates on the maps at the boundary in this setting. (Actually in their setting the complex structure and pseudoconvex hypersurface are fixed but their argument works here without change. See Eliashberg [**?**] for a concise account of these estimates.) Such uniform Lipshitz estimates imply that bubbling at the boundary cannot occur. Hence at the boundary the convergence is C^k. The result follows. q.e.d.

Proposition 5.4. *$S_0 = image(F_{t_0})$ is a smooth embedded disc in $\mathbb{R}^4$ that meets the torus $T^2(t_0)$ smoothly. Moreover, it is a minimal lagrangian surface.*

Proof. For each t, the hypersurface $r_t^{-1}(0) = \partial W_t$ contains a 2-plane distribution, denoted ρ_t, consisting of the J_t-complex lines. The intersection of ρ_t with the surface $T^2(t)$ defines on $T^2(t)$ an orientable line field, called the characteristic line field, with singularities at the complex tangent points of $T^2(t)$. The boundaries of J_t-holomorphic discs are transversal to the characteristic line field. Using the strict pseudoconvexity of ∂W_t and the fact that the boundaries of the holomorphic discs are bounded away from the complex tangent points, it follows that the angle between the boundary curve of a holomorphic disc and the characteristic line field is uniformly bounded away from zero (see [**?**] for more details). Hence the limit of the embedded boundary curves is embedded, and the limit holomorphic map, F_{t_0}, is nonsingular along the boundary, ∂D, of its domain.

By Theorem 5.3 we have a family $\{F_t : 0 \leq t \leq t_0\}$ of J_t-holomorphic maps depending continuously on t. For each t we let $\mathrm{Sing}(F_t)$ denote the number of singularities of F_t in D counted according to multiplicity. (An ordinary double point contributes one to this number.) The above argument shows that F_{t_0} is nonsingular near the boundary. All other maps of the family are nonsingular near the boundary by hypothesis. From the adjunction formula (see McDuff [**?**] for more details) it thus follows that if $0 \leq t_1, t_2 \leq t_0$, then

$$\mathrm{Sing}(F_{t_1}) = \mathrm{Sing}(F_{t_2}).$$

Since for $t < t_0$, the maps F_t are nonsingular, we have,

$$\mathrm{Sing}(F_{t_0}) = 0.$$

Therefore, F_{t_0} is nonsingular.

The last statement of the proposition follows since S_0 is a J-holomorphic curve for $J \in \mathcal{J}_0$. q.e.d.

Let $S_t \subset \mathbb{R}^4$ denote the graph of ψ_t and let

$$\begin{aligned} \pi_1 : \mathbb{R}^4 &\rightarrow \mathbb{R}^2, \\ (x_1, y_1, x_2, y_2) &\mapsto (x_1, y_1), \end{aligned}$$

denote the projection.

Lemma 5.5. *The Jacobian of the diffeomorphism* $\pi_{1|_{S_t}} : S_t \rightarrow D_1$, *computed with respect to the induced metric on* S_t, *equals* $1/\sqrt{2}$.

Proof. Consider the diffeomorphism,

$$\begin{aligned} (\pi_1)^{-1} : D_1 &\rightarrow S_t, \\ (x, y) &\mapsto (x, y, u, v), \end{aligned}$$

Under $d(\pi_1{}^{-1})$, we have

$$\begin{aligned} \tfrac{\partial}{\partial x} \mapsto X &= \tfrac{\partial}{\partial x_1} + u_x \tfrac{\partial}{\partial x_2} + v_x \tfrac{\partial}{\partial y_2}, \\ \tfrac{\partial}{\partial y} \mapsto Y &= \tfrac{\partial}{\partial y_1} + u_y \tfrac{\partial}{\partial x_2} + v_y \tfrac{\partial}{\partial y_2}. \end{aligned} \tag{5.6}$$

Let $g = g_{ij}$ be the metric on S_t induced from the euclidean metric on $\mathbb{R}^4$. Then,

$$\begin{aligned} g_{11} &= 1 + u_x{}^2 + v_x{}^2, \\ g_{12} &= u_x u_y + v_x v_y, \\ g_{22} &= 1 + u_y{}^2 + v_y{}^2. \end{aligned}$$

Using (4.1)(a) it is easy to show that $\det\ g = 2$. Let,

$$\begin{pmatrix} \tilde{X} \\ \tilde{Y} \end{pmatrix} = A \begin{pmatrix} X \\ Y \end{pmatrix} \tag{5.7}$$

be an orthonormal frame on S_t, where A is a 2×2 nonsingular matrix. This implies,

$$X \wedge Y = \det\ A^{-1}\ \tilde{X} \wedge \tilde{Y}. \tag{5.8}$$

Hence from (5.6) we have,

$$\det(d(\pi_1{}^{-1})) = \det\ A^{-1}. \tag{5.9}$$

From (5.7) it follows that, $\mathrm{Id} = AgA^t$. Thus using $\det\ g = 2$,

$$\det\ A^{-1} = (\det\ g)^{\frac{1}{2}} = \sqrt{2}. \tag{5.10}$$

Combining (5.9) and (5.10) we conclude $\det(d(\pi_{1|_{S_t}})) = \frac{1}{\sqrt{2}}$. q.e.d.

Theorem 5.6. *S_0 is the graph of a diffeomorphism*

$$\psi_{t_0} : D_1 \to D_2(t_0).$$

Proof. The surface S_t is also the image of the maps f_t and F_t. Since the maps F_{t_γ} converge in C^k to F_{t_0} it follows from the lemma that, using the induced metric on S_0, we have,

$$\det(d(\pi_{1|_{S_0}})) = \frac{1}{\sqrt{2}}.$$

In particular, $\pi_{1|_{S_0}} : S_0 \to D_1$ is a local diffeomorphism and hence a global diffeomorphism.

A similar argument shows that the projection,

$$\begin{aligned} \pi_2 : \mathbb{R}^4 &\to \mathbb{R}^2, \\ (x_1, y_1, x_2, y_2) &\mapsto (x_2, y_2), \end{aligned}$$

restricted to S_0 is a diffeomorphism $S_0 \to D_2(t_0)$. Set,

$$\psi_{t_0} = \pi_2 \circ (\pi_{1|_{S_0}})^{-1}.$$

q.e.d.

The graph of ψ_{t_0} is the surface S_0 and therefore is both minimal and lagrangian. This completes the proof of " closedness " and the proof of Theorem 5.1.

6. Uniqueness

Let $\psi = (u, v) : D_1 \to D_2$ be a minimal lagrangian diffeomorphism. Then (u, v) satisfy the equations:

$$\begin{aligned} u_x v_y - u_y v_x &= 1, \\ \sin(\pi\beta)(u_y - v_x) &= -\cos(\pi\beta)(u_x + v_y), \\ r_2(u,v) = 0 \quad &\text{if} \quad r_1(x,y) = 0, \end{aligned} \tag{6.1}$$

where β is a constant. We have already remarked at the beginning of §4 that by defining the lagrangian angle using different parallel unit $(2,0)$ forms, any value of β in (6.1) can be obtained. This is the observation made to produce a solution of the second boundary value problem for the Monge-Ampère equation. Note however that the value of β remains unchanged if both the (x, y) coordinates on the domain $\mathbb{R}^2$ and the (u, v) coordinates on the target $\mathbb{R}^2$ are rotated the same amount. Thus, up to such diagonal rotations of coordinates, given a minimal lagrangian diffeomorphism there is a unique choice of β such that the diffeomorphism is the gradient of a function. Of course if the domains are not connected or simply connected there is in general no such choice.

Brenier [**?**] proves the existence and uniqueness of a weak solution of the second boundary value problem under very general conditions on the domain that are, in particular, satisfied when the domains have smooth boundary. Hence we have:

Theorem 6.1. *If D_1 and D_2 are connected, simply connected domains with smooth boundary and equal area, then there is at most one minimal lagrangian diffeomorphism $D_1 \to D_2$ up to diagonal rotations.*

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